A note on controlling a chaotic tatonnement[1]

Venkatesh Bala[a], Mukul Majumdar[b,*], Tapan Mitra[b]

[a] Department of Economics, McGill University, Montreal, Canada
[b] Department of Economics, Cornell University, Ithaca, NY, USA

Abstract

The paper studies the possibility of 'targeting' or 'controlling' a chaotic tatonnement by suitable perturbations of the law of motion. The analysis is in the context of a parametric class of exchange economies which are shown to constitute the logistic family of dynamical systems under the tatonnement process. Computer simulations suggest that the control method is effective in attaining neighborhoods of competitive equilibria for many members of this class of economies in a decentralized manner. © 1998 Elsevier Science B.V.

1. Introduction

The mathematical model of the Walrasian tatonnement formalized by Samuelson has been extensively analyzed by a number of authors (see Arrow and Hahn (1971) for a detailed discussion). An interesting feature of the process, not always duly emphasized, is that it is an *informationally decentralized* way of obtaining market clearing prices. In the 1960s, the examples provided by Scarf (1960) and David Gale (1963) showed that even with a small number of goods and agents, there could be cycles and local instability when strong income effects were present. Subsequently, Saari (1985), Day and Pianigiani (1991) and Bala and Majumdar (1992) demonstrated that the tatonnement could exhibit

---

* Corresponding author. Tel.: +1 607 255 3640; fax: +1 607 255 2818; e-mail: mkm5@cornell.edu

[1] Part of this paper relies upon our manuscript "Controlling chaos: Some analytical results and applications to tatonnement," which was presented at the Fifth Viennese Workshop on Advances in Nonlinear Economic Dynamics held in May 1995.

'chaotic' behavior in a simple discrete time formulation with two agents and two commodities. In Section 3 of this paper, we identify a class of economies (again, with two goods and two agents) where the discrete time tatonnement process is given by the logistic family of maps $p_{(t+1)} = \mu p_t(1 - p_t)$; thus the dynamic behavior displays the full range of complexities shown by this well-studied family. This example is of independent pedagogical interest.

In the rest of the paper, we turn to the possibility of 'targeting' or 'controlling' a chaotic tatonnement by an appropriate perturbation of the law of motion in order to attain specific states; for example, fixed points or economic equilibria. The idea of such control has been explored in the physical and biological sciences (see Shinbrot et al. (1992a, 1992b, 1993)). Informally stated, the notion underlying the possibility of control is this: consider a dynamical system (X, h) where X is the *state space* and h is the *law of motion*. The law of motion h is assumed to be chaotic according to some definition. Suppose that from an initial state $x_o \in X$ a 'controller' can 'steer' the system in the first period to any element of a subset $C(x_o)$ of the state space X. Subsequently, the system is allowed to evolve under the original law of motion h. If the state of the system in some finite period T is in a prescribed neighborhood of a target state x^*, the controller has achieved his objective and stops the process. Unlike papers in the natural and biological sciences, our formulation of the control problem (described in Section 2) does not involve repeated interventions by a controller, but only a 'one-shot' one of the kind described above.

In Section 4, we employ the control technique to the chaotic tatonnement obtained in Section 3. As with the original tatonnement, the control process is informationally decentralized: an economic interpretation of it is that the controller (or auctioneer) initially announces a set of prices rather than a single one. Based upon the excess demands reported by the agents, a new set of prices is generated, with the process continuing until an 'ε-equilibrium' price is attained, at which point the process terminates. We report upon some computer experiments which show that the controller is able to attain the ε-equilibrium price quite rapidly for a number of economies where the tatonnement displays chaos.

2. Targeting: An analytical result

Consider a dynamical system (X,h) where X is a non-degenerate interval of the real line and h is a continuous map from X to X. A point $x \in X$ is a *fixed point* of h if $x=h(x)$; x is a periodic point of period $k \geq 1$ if $x=h^k(x)$ and k is the smallest positive integer with this property. In particular, a fixed point is a periodic point of period one. The pair (X,h) is a *dynamical system* where X is the *state space* and h is the *law of motion*. Now, h is said to be *topologically transitive* if for any pair of (nonempty) open intervals U and V in X there is some $j \geq 0$ such that $h^j(U) \cap V \neq \emptyset$. The relationship between topological transitivity and chaos is discussed in Devaney (1989), Banks et al. (1992) and Block and Coppel (1992). In our case since X is an interval, h is topologically transitive if and only if h is chaotic in the sense of Devaney. In particular, we know that if $X=[0,1]$, and $h(x)=4x(1-x)$ then h is chaotic in the sense of Devaney (1989, page 50, Example 8.9).

Suppose from an initial state x_o that we wish to 'attain' or, at least, visit a neighborhood of a 'target' state x^*. If the dynamical system (X,h) is chaotic, it is not easy to verify whether this is possible in a finite number of periods. Now suppose that a 'controller' has the ability to achieve 'small' perturbations of the laws of motion h; under what conditions can we say that, if we choose an arbitrarily small open neighborhood $N_{x^*} \subset X$ containing x^*, the trajectory from an initial x_o will visit N_{x^*} (i.e. $h^j(x_o)$ will be in N_{x^*} for a finite j)? In an economic context, repeated and fairly 'refined' interventions may not be feasible. So we consider the possibility of a 'one-shot intervention.' Let C be a class of functions from X to X containing a particular law of motion h, from which the controller can select in the initial period. After choosing some $\hat{h}$ from C, the system is 'left alone' to be governed by the law of motion h and stopped, if the system attains a preassigned neighborhood in finite time; in other words, from any $x_o \in X$ we consider a sequence of states evolving according to the following rule:

$$\hat{x}_1 = \hat{h}(x_0), \quad \hat{h} \in \mathbf{C}$$
$$x_2 = h(\hat{x}_1)$$
$$\dots \tag{1}$$
$$x_{t+1} = h(x_t), \quad \text{for all } t \geq 2$$

Then our problem of 'visiting' N_{x^*} is reduced to: does there exist some $\hat{h} \in C$ such that $\{x_0, \hat{x}_1, x_2, \dots\}$ obtained according to Eq. (1) has the property that $x_T \in N_{x^*}$ for some finite T? To answer this question, consider the set

$$\mathbf{C}(x_0) = \{y \in X \mid y = g(x_0), g \in \mathbf{C}\}. \tag{2}$$

The set $\mathbf{C}(x_o)$ is the set of all possible states that the controller can attain in period one from the initial state x_o. The question of 'visiting' any open set containing the target state x^* in finite time is made precise in the following:

Proposition 1 *Suppose X is a non-degenerate interval on the real line and that (a) h: $X \to X$ is a continuous map which is topologically transitive and (b) $C(x_o)$ contains a nonempty open subset $L(x_o)$ of X. Given an open interval N_{x^*} containing a target state x^*, there exists $\hat{h} \in C$ and an integer $T \geq 0$ such that $h^T (\hat{h}(x_o))$ lies in N_{x^*}.*

Proof: Since h is topologically transitive and $L(x_o)$ is open, there is some $T \geq 0$ such that $h^T(L(x_o)) \cap N_{x^*} \neq \emptyset$. Let $\hat{x} \in L(x_o)$ be such that $h^T(\hat{x} \in N_{x^*}$. Since $L(x_o) \subset C(x_o)$ by assumption, $\hat{x} = \hat{h}(x_o)$ for some $\hat{h} \in C$. The result follows.

3. Chaotic tatonnement

Recall that in the Walras–Samuelson tatonnement process the auctioneer announces an initial price and then adjusts the price upward or downward based upon the agents' reported excess demands, with the process continuing in the same manner until an equilibrium is reached. To underscore the economic relevance of the controlled tatonnement process introduced in the Section 4, we emphasize that the tatonnement is an *informationally decentralized* method by which a competitive equilibrium price may

be attained. Note that at any point of time, the auctioneer neither knows the excess demand functions nor the individual characteristics of the agents who generate their excess demand functions, that is, the utility functions and endowments of the agents. (If he did, he could directly solve for the competitive equilibrium price). Likewise, no agent knows the characteristics of other agents or even the price adjustment rule followed by the auctioneer. Thus the tatonnement is entirely anonymous (formally, the mechanism is 'privacy-preserving' in the sense of Hurwicz (1986)).

In this section we show that when the discrete-time tatonnement is applied to a parametric family of two-good (x_1 and x_2), two-consumer exchange economies, the induced dynamical system constitutes the logistic family

$$p_{t+1} = \mu p_t(1 - p_t).$$ (3)

where $p_t \in [0,1]$ is the price of good 1 in period t and the price of good 2 is given by 1. As the logistic family exhibits an enormous variety of complicated behavior, this example is of independent pedagogical interest.

We now give details of the construction. For this class of economies, the rate of price adjustment in the tatonnement is fixed. The consumers have simple preferences give by the Leontief type (for the first consumer) and the quasi-linear type (for the other consumer). The parameter μ relates to the preferences of the two consumers. Consumer 1 has the utility function

$$u^1(x_1, x_2) = \min\{f(x_1), \alpha x_2\}.$$ (4)

where

$$f(x_1) = \begin{cases} x_1 - x_1^2, & \text{if } 0 \leq x_1 \leq 1/2; \\ 1/4, & \text{if } x_1 > 1/2. \end{cases}$$ (5)

and $\alpha \in [1/3, 1/2]$ is a preference parameter. Furthermore, the consumer's initial endowment is $\omega^1 = (1,0)$. His demand function for x_1 is then given by:

$$x_1^1(p) = \begin{cases} \alpha p, & \text{if } 0 \leq p \leq 1/(2\alpha); \\ 1/2, & \text{if } p > 1/(2\alpha). \end{cases}$$ (6)

where p denotes the price of good 1. This may be shown as follows: we consider the two equations

$$\alpha x_2 = x_1 - x_1^2, \quad px_1 + x_2 = p.$$ (7)

which hold when $p \in [0, 1/(2\alpha)]$. Substituting for x_2 we get the quadratic

$$x_1^2 - (\alpha p + 1)x_1 + \alpha p = 0.$$

which has the solutions αp and 1. Thus, $x_1(p) = \alpha p$ and $x_2(p) = p(1-\alpha p) \geq 0$ constitutes a solution to the eqautions in Eq. (7) when $p \in [0, 1/(2\alpha)]$. Note also that $x_1(p) = \alpha p \leq 1/2$ so that $f(x_1(p)) = x_1(p) - x_1(p)^2$ and

$$\alpha x_2(p) = f(x_1(p)), \quad px_1(p) + x_2(p) = p.$$ (8)

Finally, for $p > 1/(2\alpha)$, we have $x_1(p) = 1/2$ and $x_2(p) = 1/(4\alpha)$.

The utility function of consumer 2 is of the quasi-linear form:

$$u^2(x_1, x_2) = g(x_1) + x_2. \tag{9}$$

where

$$g(x_1) = \begin{cases} -\beta^{1/2}\left(\frac{2}{3}\right)(1 - x_1)^{3/2}, & \text{if } 0 \le x_1 \le 1; \\ 0, & \text{if } x_1 > 1. \end{cases} \tag{10}$$

where $\beta \in [3/2, 2]$. Furthermore, his initial endowment is $\omega^2 = (0, 1)$. The consumer's demand for x_1 can then be computed as

$$x_1^2(p) = 1 - \frac{p^2}{\beta}, \tag{11}$$

This is shown as follows: consider the equations

$$g'(x_1) = p, \ px_1 + x_2 = 1. \tag{12}$$

The solution to Eq. (12) also solves the consumer's utility maximization problem. For $x_1 \in [0, 1]$, we have:

$$g'(x_1) = -\beta^{1/2}\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)(1 - x_1)^{1/2}(-1) = \beta^{1/2}(1 - x_1)^{1/2}. \tag{13}$$

Putting $g'(x_1) = p$ we obtain $x_1^2(p) = 1 - (p^2/\beta)$. For $p \in [0, 1]$, (using $\beta \ge 0$), we have $x_1^2(p) \ge 0$, and $x_2^2(p) = 1 - px_1^2(p) = 1 - p + p^3/\beta$ which is also non-negative. Thus, for $p \in [0, 1]$, $x_1^2(p) = 1 - (p^2/\beta)$ and $x_2^2(p) = 1 - p + p^3/\beta$ solves consumer 2's maximization problem; in particular, Eq. (11) holds for $p \in [0, 1]$.

The main idea employed here is that since consumer 1 is necessarily a seller of good 1, his preferences can be made to lead to an increase in the demand for x_1 with an increase in the price of x_1 by allowing low substitution possibilities. The Leontief-type (but without fixed coefficients) preferences that we use have the further advantage that the demand for x_1 is in fact *linear* in p. Consumer 2 is necessarily a buyer of good 1 and his demand for x_1 will decrease in p in the usual case. By choosing his utility function of the particular quasi-linear type that we use, we ensure that for small p it decreases in p at a very low rate and for p close to 1 it decreases in p at a relatively high rate (i.e. as the square of the price). Adding the two effects and subtracting the total supply of 1, we will generate the inverted U-shaped excess demand function. The quadratic family can now be generated by fixing θ as high enough and relating the taste parameter α and β appropriately. Formally, from Eqs. (6) and (11) we get the excess demand for good 1 as:

$$z(p) = \left\{1 - \frac{p^2}{\beta} + \alpha p\right\} - 1 = \alpha p - \frac{p^2}{\beta}. \tag{14}$$

The discrete-time tatonnement process has the form

$$p_{t+1} = p_t + \theta z(p_t) \tag{15}$$

where p_t is the price of good 1 at time t and $\theta > 0$ is a parameter depending upon the market rate of adjustment of the price. We shall apply this dynamic to a class of economies parameterized by a single parameter ξ. The market rate of adjustment will be

fixed at $\theta=6$. Regarding the taste parameters, α and β, the choices will be as follows: for any $\xi \in [1/3,1/2]$, we shall set

$$\alpha = \xi, \ \beta = \frac{6}{1 + 6\xi}. \tag{16}$$

Note that $\alpha \in [1/3,1/2]$ and $\beta \in [3/2,2]$. Thus the excess demand for the economy ξ is

$$z(p) = \xi p - \frac{(1 + 6\xi)p^2}{6} \tag{17}$$

and, setting $\theta=6$, the tatonnement equation for the economy ξ is

$$p_{t+1} = p_t + 6\left\{ \xi p_t - \frac{(1 + 6\xi)p_t^2}{6} \right\} = (1 + 6\xi)p_t(1 - p_t). \tag{18}$$

Finally, denoting $(1+6\xi)$ by μ we get the logistic equation:

$$p_{t+1} = h_\mu(p_t) \equiv \mu p_t(1 - p_t) \tag{19}$$

as the tatonnement dynamic. Furthermore, as $\xi \in [1/3,1/2]$, we get $\mu \in [3,4]$. Now all the analysis developed for the quadratic family can be applied to this family of economies indexed by the parameter ξ (or equivalently, μ).

4. Computer experiments

In this section, we conduct computer simulations on the class of economies considered in Section 3. We can employ the technique of Section 2 to locate a neighborhood of an interior fixed point of h_μ which corresponds to a competitive equilibrium for the underlying economy, whenever the tatonnement process for the economy exhibits chaos. We start with the system $p_{t+1} = h_4(p_t) = 4p_t(1 - p_t)$. Fix $\bar{\mu} \in (3,4)$, and let K denote the interval $[\bar{\mu}, 4]$. Denote **C** as the set of functions $\{h\mu\}$ for $\mu \in K$. For any $p_o \in (0,1)$ we consider the set

$$\mathbf{C}(p_o) = \{y \in [0, 1] \mid y = h_\mu(p_o), \mu \in K\}. \tag{20}$$

Clearly, since p_o is strictly between 0 and 1, the set $\mathbf{C}(p_o) \subset [0,1]$ contains an open interval $L(p_o)$ of values by the intermediate value theorem. Thus, by topological transitivity of h_4, the sequence of iterates of h_4 of $L(p_o)$ will eventually approach any neighborhood of a fixed point p^* of h_4.

An economic interpretation of the above procedure is that the 'auctioneer,' rather than just choosing an initial price, chooses a 'large' number of initial prices, and for each of those prices, obtains the value of excess demand from the agents in the economy. From this a new set of possible prices is computed by applying the tatonnement individually to each of the original prices. In subsequent periods this procedure is repeated until for some price in the set of prices, the absolute value of excess demand is at most ϵ, where $\epsilon > 0$ has been previously fixed. (We shall refer to such a price as an ϵ-equilibrium price). If the tatonnement applied to the excess demand is topologically transitive, then within finite time the above procedure will generate an ϵ-equilibrium price. We emphasize that just
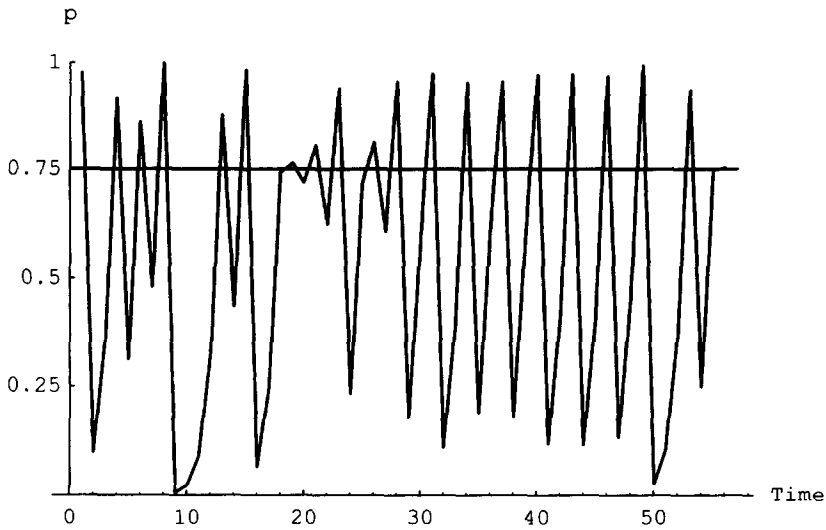
Fig. 1. Controlling chaos ($\mu=4$)

like the original tatonnement process, this procedure is also informationally decentralized, in the sense discussed in Section 3.

To implement the method on a computer, we need to discretize the space. Consequently, we choose Q points $\{\mu_i\}_{i=1}^{Q}$ in $K=[\bar{\mu},4]$, where Q is 'large.' The parameter μ_i is referred to as a *control*. For each control μ_i, we compute $\hat{p}_1^i = h_{\mu_i}(p_o)$, and then $p_2^i = h_4(\hat{p}_1^i)$, $p_3^i = h_4(p_2^i)$ and so on. For fixed $\epsilon>0$ we check if $| h_4(p_t^i) - p_t^i |\leq \theta\epsilon$ (recall from Section 3 that $\theta=6$ is the market adjustment rate). If this inequality is met, the absolute value of excess demand for the economy corresponding to h_4 is at most ϵ.

In our computations, we have chosen a tolerance of $\varepsilon=0.001$, $Q=50$ equally spaced points from the interval [3.9,4] and a p_0 at random from the [0,1] interval. For each μ_i in the grid, we stop the iterations after at most n periods (here $n=100$) if no 'ε fixed point' has been reached. Fig. 1 summarizes the results of a simulation of the control method for the dynamical system h_4. The initial price is $p_0=0.562$ approximately, and the control value is $\mu_i=3.96$. In order to better indicate the quality of the targeting process, we have drawn a horizontal line at the equilibrium price $p^*=0.75$. As can be seen from the figure, an ε-equilibrium price is attained at $t=57$. Similar results were obtained for a number of other values of the control μ_i.

While the method seems to work, we note one difficulty with this approach: it is possible that the process may come close to the fixed point of h_4 at $p=0$ which is economically inappropriate. In fact, a neighborhood around $p=0$ was found far more frequently in our simulations. Fig. 2 illustrates one such instance, for the control value $\mu_i=3.95$. Here the process has stopped at $t=35$ where the price is very close to the fixed point at 0. We note that this difficulty is somewhat special to the case $\mu=4$; its occurrence can be ruled out when $\mu<4$. The formal argument is long and tedious and the interested reader is referred to Bala et al. (1995).
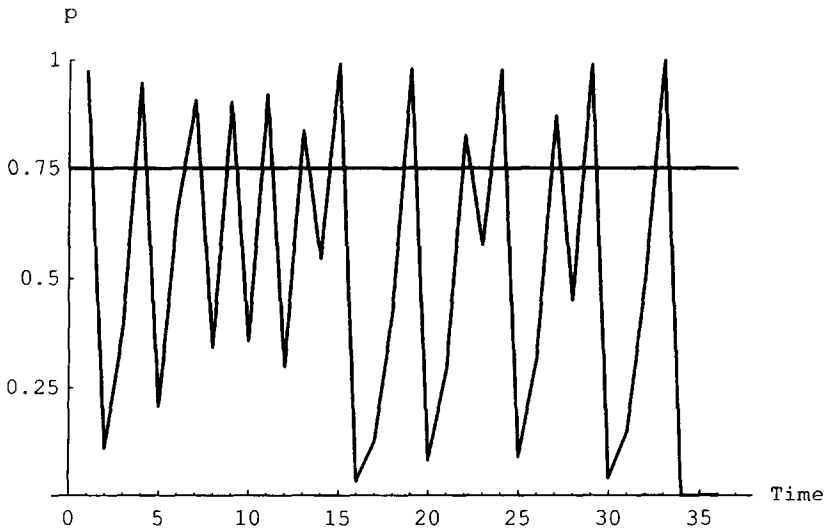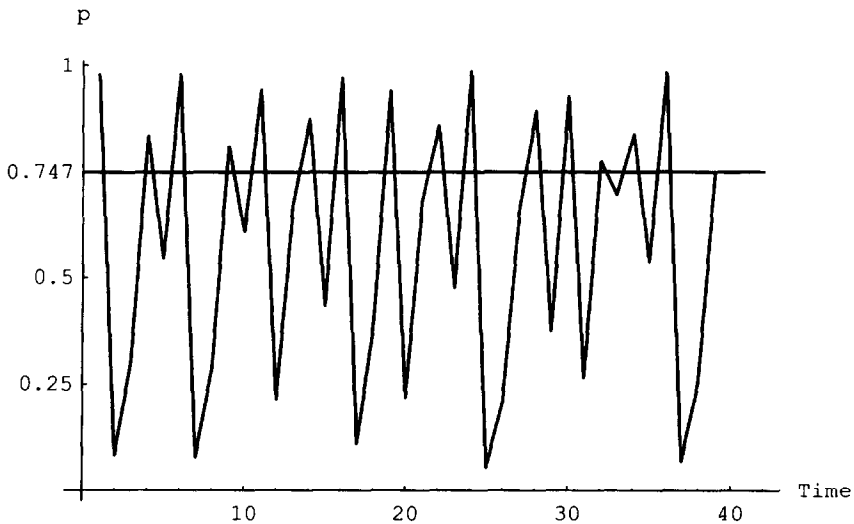
P

Fig. 2. Controlling chaos (μ=4)

P

Fig. 3. Controlling chaos

Next, we report our computer simulations for $\tilde{\mu} = 3.95$. The interior fixed point of $h_{\tilde{\mu}}$ is approximately at p^*=0.746835. A total of 50 grid points between 3.9 and 4 were chosen along with an initial value p_o of approximately 0.5174. The value of ε=0.001 is as before. Fig. 3 shows the control process when the first period control value is μ_i=3.918. As before, a horizontal line has been drawn at the equilibrium value. As can be seen, the process halts at t=41 within the ϵ-tolerance of p^*. Fig. 4 shows another simulation when μ_i=3.964. The system stops at t=78 within 0.001 of p^*. We also note that in all the
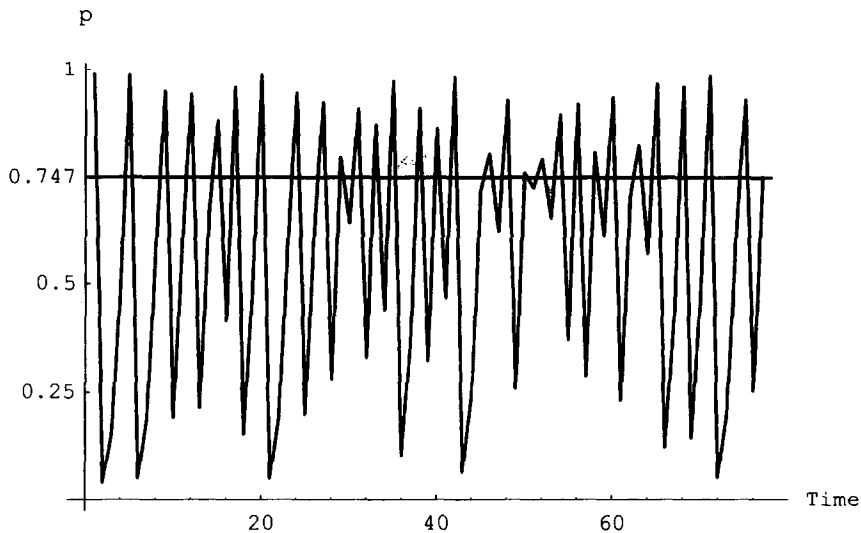
Fig. 4. Controlling chaos

simulations, the system never approached the fixed point around 0, as will always be the case when $\tilde{\mu} < 4$.

5. Conclusions

It is well known that the tatonnement process applied to even simple economies with small numbers of agents and commodities can exhibit chaos. In this paper, we demonstrate how a Walrasian auctioneer can 'control' a chaotic tatonnement process to lead to an arbitrarily small neighborhood of a competitive equilibrium price. We begin by constructing a parametric class of exchange economies which yield the logistic family of dynamical systems when the tatonnement is applied to them. Our computer experiments suggest that the control method described in this paper works quite efficiently to locate a given neighborhood of competitive equilibrium for a large subset of the parametric class of economies.

References

Arrow, K.J., Hahn, F., 1971. General Competitive Analysis, Holden Day, San Francisco.
Bala, V., Majumdar, M., 1992. Chaotic tatonnement, Economic Theory 2, 437–445.
Bala, V., Majumdar, M., Mitra, T., 1995. Controlling chaos: Some analytical results and applications to tatonnement, presented at the Fifth Viennese Workshop on Advances in Nonlinear Economic Dynamics held in May 1995, and circulated as CAE Working Paper 96-17, Center for Analytic Economics, Cornell University, Ithaca, NY.
Banks, J., Brooks, J., Cairns, G., Davis, G., Stacey, P., 1992. On Devaney's definition of chaos, American Mathematical Monthly 99, 332–334.

Block, L.S., Coppel, W.A., 1992. Dynamics in One Dimension, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg.

Day, R.H., Pianigiani, G., 1991. Statistical dynamics and economics, Journal of Economic Behavior and Organization 16, 37–84.

Devaney, R.L., 1989. An Introduction to Chaotic Dynamical Systems, 2nd edn., Addison Wesley, New York.

Gale, D., 1963. A note on the global instability of competitive equilibrium, Naval Research Logistics Quaterly 10, 81–87.

Hurwicz, L., 1986. On informational decentralization and efficiency in resource allocation mechanisms, in: Reiter, S. (Ed.), Studies in Mathematical Economics, 25, MAA Studies in Mathematics, The Mathematical Association of America, pp. 238–250.

Saari, D.G., Iterative price mechanisms, Econometrica 53, 1117–1132.

Scarf, H., 1960. Some examples of global instability of competitive equilibrium, International Economic Review 1, 157–172.

Shinbrot, T., Ott, E., Grebogi, C., Yorke, J.A., 1992. Using chaos to direct orbits to targets in systems describable by a one-dimensional map, Physical Review, A 45, 4165–4168.

Shinbrot, T., Ditto, W., Grebogi, C., Ott, E., Spano, M., Yorke, J.A., 1992. Using the sensitive dependence of chaos (the Butterfly effect) to direct trajectories in an experimental chaotic system, Physical Review Letters 68, 2863–2866.

Shinbrot, T., 1993. Chaos: Unpredictable yet controllable, Nonlinear Science Today 3, 1–8.